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Reidemeister torsion, the Alexander polynomial and U(1, 1) Chern–Simons theory

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We show that the U(1, 1) (super) Chern–Simons theory is one-loop exact. This provides a direct proof of the relation between the Alexander polynomial and analytic and Reidemeister torsions. We then compute explicitly the torsions of Lens spaces and Seifert manifolds using surgery and the *S* and *T* matrices of the U(1, 1) Wess–Zumino–Witten model recently determined, with complete agreement with known results. U(1, 1) quantum field theories and the Alexander polynomial thus provide "toy" models with a non-trivial topological content, where all ideas put forward by Witten for SU(2) and the Jones polynomial can be explicitly checked, at finite *k*. Some simple but presumably generic aspects of non-compact groups, like the modified relation between Chern–Simons and Wess–Zumino–Witten theories, are also illustrated. We comment on the closely related case of GL(1, 1).

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1. Introduction

Quantum field theory plays an increasing role in the study of topological invariants in low dimensions [1]. In his seminal paper [2] Witten has shown how SU(2) Chern–Simons theory gives rise to the Jones polynomial for links in S^3, to new three-manifold invariants, and to new "Jones" invariants of links in manifolds. Subsequent developments have taken place in several directions. In particular, three-manifold invariants have been defined more rigorously [3–5] and their interrelations studied [6]. Also three-manifold invariants have been explicitly obtained via surgery and the *S* and *T* matrices of the Wess–Zumino–Witten theories, either numerically [7] or analytically [8]. Of particular interest in that case has been the large-*k* expansion [2,9], in which the analytic torsion [10], equal to the Reidemeister torsion [11,12], appears. Of course the complete three-

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manifold invariants in the SU(2) case contain more topological information than the torsion.

Following these recent works on the Jones polynomial, the older Alexander–Conway [#1] polynomial [14,15] has been reconsidered and partly put in this more modern perspective [16,17]. It appears interesting to do so in order, for instance, to gain more insight on the topological meaning of the Jones polynomial itself, or to have at hand a case where simple computations can be made and the whole construction tested. In ref. [17], the S and T matrices of the U(1, 1) Wess–Zumino–Witten model have been computed, and it has been shown how surgery allows one to compute the multivariable Alexander–Conway polynomial of links in S^3. The purpose of this letter is to study invariants of manifolds and their relations with torsion.

Relations between torsion and the Alexander invariant are well known: in ref. [18] Turaev, following previous work by Milnor [19], has established the equivalence of the Alexander polynomial of links in three-manifolds and the Reidemeister torsion of the complements of those links as well as the torsion of the manifolds themselves. Recalling the above mentioned equality of Reidemeister and analytic torsion, we complete here the relations of these three mathematical objects by showing in section 2 that the Chern–Simons theory based on U(1, 1) is one-loop exact, thus providing a direct relationship between the Alexander polynomial and analytic torsion.

We then compute in section 3 Alexander polynomials of links in a variety of Lens spaces and Seifert manifolds using the S and T matrices of the U(1, 1) Wess–Zumino–Witten model, and extract from them Reidemeister torsions. Such computations are done in the large-k limit for, e.g., SU(2) but hold for finite k here due to the one-loop exactness. Section 4 deals with manifolds with an infinite H_1. Some conclusions are gathered in section 5.

2. One-loop exactness of U(1, 1) Chern–Simons

In all this paper we use notations of ref. [16]. Recall the expression of the "classical" U(1, 1) Chern–Simons action in components,

$$S_{\rm cl}=\frac{k}{2\pi}\int \epsilon_{\mu\nu\rho}(A^\mu_N\partial^\nu A^\rho_E+A^\mu_\psi(\partial^\nu+A^\nu_E)A^\rho_{\psi^+})\ {\rm d}^3x\,, \tag{2.1}$$

where we have set

$$A^\mu=A^\mu_E N+A^\mu_N E+A^\mu_\psi\psi^+-A^\mu_{\psi^+}\psi\,. \tag{2.2}$$

[#1] By Alexander polynomial we refer to the original invariant defined up to powers of the variable t. By Alexander–Conway polynomial we refer to its normalized version. See ref. [13] and references therein.

Recall that the generators satisfy the relations

$$\{\psi, \psi^+\} = E\,, \qquad [N, \psi^+] = \psi^+\,, \qquad [N, \psi] = -\psi\,, \qquad E \text{ is central}\,. \tag{2.3}$$

We consider first purely bosonic flat connections, determined by

$$F_E^{\mu\nu} = \partial^\mu A_E^\nu - \partial^\nu A_E^\mu = 0\,, \qquad F_N^{\mu\nu} = \partial^\mu A_N^\nu - \partial^\nu A_N^\mu = 0\,. \tag{2.4}$$

They are labelled by maps from $H_1(\mathscr{M})$ into $U(1)_N \times U(1)_E$ (H_1 arises instead of Π_1, $U(1)$ being abelian). Since the bosonic generators commute among themselves, flat connections are always reducible.

The one-loop approximation to three-manifold invariants is obtained as in the standard case. Call $A^{(\alpha)}$ a complete set of gauge equivalent flat connections on the manifold $\mathscr{M}$. Assume that there is a finite number of them, and for the moment that none of them are reducible. Expanding the fields around $A^{(\alpha)}$ we get a quadratic action, to be supplemented by proper gauge fixing terms, after a metric is picked on $\mathscr{M}$. The A_E and A_N degrees of freedom commute with $A^{(\alpha)}$ and the resulting integrations do not depend on it. We get in the notations of ref. [2]

$$(\det(\Delta))^2 / |\det(L_-)| \tag{2.5}$$

[each acting on "one-colored" objects, as opposed to the three colors for SU(2)]. For A_ψ and A_{ψ^+} we get, due to statistics, the inverse of the above result, but without absolute value and with the operators Δ and L_- being twisted by the $A_E^{(\alpha)}$ part of the flat connection (the $A_N^{(\alpha)}$ part is central, the generator E commuting with the entire algebra). Introducing the Chern–Simons invariant $S_{\text{cl}}^{(\alpha)}$ of the flat connection we get

$$\mathscr{Z}(\mathscr{M}) = \frac{(\det(\Delta))^2}{|\det(L_-)|^2} \sum_\alpha e^{iS_{\text{cl}}^{(\alpha)}} \frac{\det(L_-^{(\alpha)})}{(\det(\Delta^{(\alpha)}))^2}\,. \tag{2.6}$$

The phase of $\det(L_-^{(\alpha)})$ is given, as in the $U(1)_N$ case, by

$$\frac{i}{4\pi} \int \epsilon_{\mu\nu\rho} A_E^\mu \partial^\nu A_E^\rho\,. \tag{2.7}$$

This term just combines with the classical Chern–Simons action to incorporate the appropriate quantum corrections [16], which amounts to replacing

$$A_N \to A_{\tilde{N}} = A_N + \frac{1}{2k} A_E\,. \tag{2.8}$$

Up to the prefactor outside the sum, the three-manifold framing and spectral flow contribution [7], we thus get for each α the inverse of the analytic torsion (instead of its square root in the SU(2) case) as defined in refs. [10,2]. Owing to refs. [10–12] this analytic torsion coincides with the "algebraic" Reidemeister torsion as defined in ref. [20] and weighted by the exponential of the Chern–Simons action. The inverse definition is also used in the literature, as in refs.

[18] and [19], and this is what we shall do here. For clarity we introduce the name [18] of Milnor torsion so (2.6) becomes

$$\mathscr{Z} \propto \frac{1}{\tau_{\text{Mil}}^{(\text{trivial})}} \sum \tau_{\text{Mil}}$$

(and depending on the way this torsion is defined, we use the label analytic or algebraic). It is interesting to notice that the partition function (2.6) depends on the global framing of the manifold through the phase of $\det(L_-^{(0)})$. This dependence when interpreted from a two-dimensional point of view corresponds to a central charge $c=-2$. On the other hand, the U(1, 1) WZW model has $c=0$. The naive correspondence between Chern–Simons and WZW theories, well known in the compact case, does not hold here, illustrating ref. [21] [the U(1, 1) WZW model is still expected to give rise to topological invariants since such invariants can be shown to make sense from a pure two-dimensional conformal field theory point of view [4,13], although the necessary axiomatics has presumably to be modified]. Fortunately the correct correspondence is easy to establish, and does not spoil any of the results in refs. [16,17]. One simply has to change the coupling of one of the free bosons in the free field representation of the U(1, 1) WZW model, and at the same time consider it as antiholomorphic. The S and T matrices that follow are simply deduced from the ones in ref. [17] by $S\to -S$, $T\to \exp(i\pi/3)T$, giving an apparent central charge $c=-2$. All other results are unchanged.

Equation (2.6) is actually one-loop exact. This can be shown using a strategy similar to ref. [22]. For simplicity forget for a while gauge fixing. We compute then the partition function of some three-manifold by first performing the functional integration over A_N^μ, which produces a delta function

$$\delta(\epsilon_{\mu\nu\rho}\partial^\nu A_E^\rho)\,, \tag{2.9}$$

so the integral over the variable A_E is restricted to A_E flat connections $F_E^{\mu\nu}=0$. Assume as before that there is a finite number of them, called $A_E^{(\alpha)}$. For each such connection the integral over A_ψ and A_{ψ^+} is a Gaussian integral, so we get the ratio of the determinant of the kinetic operator twisted by $A_E^{(\alpha)}$ over the absolute value of the same determinant but untwisted, where the denominator originates in the Jacobian for the constraint $F_E^{\mu\nu}=0$. Now let us take gauge fixing into account. Infinitesimal gauge transformations can be written

$$\begin{aligned} \delta A_E^\mu &= \partial^\mu \omega_E\,,\\ \delta A_N^\mu &= \partial^\mu \omega_N + A_{\psi^+}^\mu \omega_\psi + A_\psi^\mu \omega_{\psi^+}\,,\\ \delta A_\psi^\mu &= \partial^\mu \omega_\psi + A_E^\mu \omega_\psi\,,\\ \delta A_{\psi^+}^\mu &= \partial^\mu \omega_{\psi^+} - A_E^\mu \omega_{\psi^+}\,. \end{aligned} \tag{2.10}$$

We choose the gauge fixing conditions

$$\partial^\mu A^\mu_E = \partial^\mu A^\mu_N = D^\mu_{(E)} A^\mu_\psi = D^\mu_{(E)} A^\mu_{\psi^+} = 0 , \tag{2.11}$$

where

$$D^\mu_{(E)} A^\mu_\psi = \partial^\mu A^\mu_\psi + A^\mu_E A^\mu_\psi , \qquad D^\mu_{(E)} A^\mu_{\psi^+} = \partial^\mu A^\mu_{\psi^+} - A^\mu_E A^\mu_{\psi^+} . \tag{2.12}$$

These conditions are implemented by adding a pair of bosonic and a pair of fermionic Lagrange multipliers to the action: ϕ^E, ϕ^N, χ^ψ, χ^{ψ^+}. We also introduce a pair of ghosts b, c for each species to restore the correct integration measure. The ghost action contains the following couplings:

$$\begin{array}{llll} b^E\partial^\mu\partial^\mu c^E , & 0 , & b^\psi(\partial^\mu c^E)A^\mu_\psi , & -b^{\psi^+}(\partial^\mu c^E)A^\mu_{\psi^+} , \\ 0 , & b^N\partial^\mu\partial^\mu c^N , & 0 , & 0 , \\ 0 , & b^N\partial^\mu(A^\mu_{\psi^+}c^\psi) , & b^\psi D^\mu_{(E)}D^\mu_{(E)}c^\psi , & 0 , \\ 0 , & b^N\partial^\mu(A^\mu_\psi c^{\psi^+}) , & 0 , & b^{\psi^+}D^\mu_{(E)}D^\mu_{(E)}c^{\psi^+} . \end{array} \tag{2.13}$$

Integration over A_N produces again the condition (2.9). However, one also has to integrate over the Lagrange parameter ϕ^E, ϕ^N so the complete Jacobian produces in the denominator the absolute value of the determinant of untwisted L_-. Integration over A_E is now restricted to the sum over flat connections $A^{(\alpha)}_E$. We also integrate over χ^ψ, χ^{ψ^+} to ensure gauge conditions for A_ψ, A_{ψ^+}. We can then integrate over the fermionic ghost b^E to produce a first determinant of the Laplacian Δ together with a δ function that restricts to zero modes of c^E with respect to $\partial^\mu\partial^\mu$. We have therefore $\partial^\mu c^E = 0$. Now the integral over the bosonic ghost b^ψ produces the inverse of the determinant of the twisted Laplacian and restricts to zero modes of c^ψ with respect to $D^\mu_{(E)}D^\mu_{(E)}$, so we have $D^\mu_{(E)}c^\psi = 0$. The term $b^N\partial^\mu(A^\mu_{\psi^+}c^\psi)$ then vanishes identically thanks to $D^\mu_{(E)}c^\psi = 0$ and the gauge fixing condition on $A^\mu_{\psi^+}$. Similar results are obtained for integration over b^{ψ^+}. All non-diagonal couplings thus disappear and we can finally integrate over b^N. The couplings (2.13) can therefore be reduced to the diagonal, from which the correct ratio of squares of determinants of the Laplacian is immediately obtained. Having gotten rid of the ghosts we integrate over all remaining variables to produce the determinant of the twisted L_- in the numerator.

Of course, integrations over A^μ_N and A^μ_E must be performed independently for each $U(1)_E$ and $U(1)_N$ bundle over the manifold. Recall that U(1) bundles are classified by maps from $H_1(\mathcal{M})$ into U(1). Also, when the bundle is non-trivial, we must take into account the correct form of the Chern–Simons action in the manifold $\mathcal{M}$. Following ref. [23] it is easy to show that, for a given bundle and its pair of A_E, A_N flat connections, the correct action for a gauge field A is the sum of the action of the flat connection $A^{(\alpha)}$ and the "naive" action of the difference $a = A - A^{(\alpha)}$. We thus recover the $S_{\rm cl}$ phase factors of (2.6). We can then integrate freely over a to recover the preceding result in each bundle. Summing over all the bundles restores also the sum over A_N flat connections that was missing so far, so we recover exactly (2.6).

Our model behaves much like the super ISU(2) model in ref. [22]. In the latter reference, however, the equivalent of flat connections $F_E^{\mu\nu}=0$ involves also a quadratic term so the two determinants in (2.6) would both be twisted, and cancel each other, giving a result simply equal to a phase for each α, that sums up to the SU(2) Casson invariant. Our model also bears some resemblance with 2+1 dimensional gravity, whose partition function would write [22] $\mathscr{Z} \propto \sum (\tau_{\text{Mil}})^{-1}$. See also ref. [24].

The formula (2.6) requires corrections for possibly non-trivial cohomologies $H^0_{\mathscr{M}}(A^{(\alpha)})$ and $H^1_{\mathscr{M}}(A^{(\alpha)})$. According to refs. [7] and [25] a non-trivial $H^0_{\mathscr{M}}(A^{(\alpha)})$ requires an extra factor of $1/\text{Vol}(H)$, where H is the subgroup of U(1, 1) commuting with all the holonomies of the connection $A^{(\alpha)}$. The cohomology $H^0_{\mathscr{M}}(A^{(\alpha)})$ is always non-trivial for a U(1, 1) connection. Generally $H=\text{U}(1)_E \times \text{U}(1)_N$, for which [17]

$$1/\text{Vol}(H)=V=1/2k\,. \tag{2.14}$$

However, if the holonomy is a subgroup of $\text{U}(1)_E$ (i.e., if $A_E^{(\alpha)}=0$ in a certain gauge, which we also call pure A_N flat connections), then $H=\text{U}(1,1)$ and $\text{Vol}(\text{U}(1,1))$ is equal to zero [17]. We thus get a diverging three-manifold invariant that can nevertheless be compared with the invariant of a reference manifold, say S^3. For A_N flat connections, the determinants (after proper subtraction of zero modes) of twisted and untwisted operators coincide in (2.6) so each of them contributes by a factor unity, as the Chern–Simons action vanishes. Other flat connections have a finite contribution, and thus are "unobservable" in that case (but see later). Therefore we simply count maps from $H_1(\mathscr{M})$ into $\text{U}(1)_E$ with the result

$$\mathscr{Z}(\mathscr{M})/\mathscr{Z}(S^3)=\text{order}(H_1(\mathscr{M}))\,, \tag{2.15}$$

where order = card here. This holds up to a phase factor encoding the global framing of the manifold [17], and as long as the H_1 of the manifold is finite. If it is infinite, there is a continuous set of maps into $\text{U}(1)_E$ and the sum over flat connections has to be replaced by an integral (see next paragraph). The final result is presumably finite, so (2.15) should still hold if we define the order to be zero when the cardinal is infinite:

$$\text{order}=\text{card if card}<\infty\,, \quad \text{order}=0 \text{ otherwise}\,. \tag{2.16}$$

The result (2.15) then agrees with the numerical value of the Milnor torsion of $\mathscr{M}$ for the trivial representation of its Π_1 (see ref. [18]). Recall that in ref. [17] we recovered (2.15) by surgery computations using S and T matrices for the U(1, 1) WZW model. Taking into account the modified correspondence between Chern–Simons and WZW theories gives an analogous result, up to a change of framing dependence $c=2 \to c=-2$.

A non-trivial $H^1_{\mathcal{M}}(A^{(\alpha)})$ signals the existence of a continuous family of flat connections, parametrized by a moduli space. This means that the sum in (2.6) has to be replaced by an integral over that space. $H^1_{\mathcal{M}}(A^{(\alpha)})$ takes values in the adjoint representation of U(1, 1). Its bosonic part $\mathrm{Bos}(H^1_{\mathcal{M}}(A^{(\alpha)}))$ is equal to the tensor square of the ordinary cohomology of $\mathcal{M}$ over the real numbers:

$$\mathrm{Bos}(H^1_{\mathcal{M}}(A^{(\alpha)}))=H^1_{\mathcal{M}}(R)\times H^1_{\mathcal{M}}(R) . \tag{2.17}$$

A particular example of a non-trivial $\mathrm{Bos}(H^1_{\mathcal{M}}(A^{(\alpha)}))$ is provided by the manifold $S^2\times S^1$ and will be considered later.

Let us finally discuss the fermionic flat connections. First one can easily show that for any finite subgroup of U(1, 1), each element is conjugate to a purely bosonic one. In the case of a Lens space, for instance, for which $\Pi_i = Z_p$, there are therefore only bosonic flat connections (up to gauge equivalence) and the preceding discussion suffices. In general, if a flat connection has a fermionic component, the latter cannot be isolated, which implies a non-trivial fermionic part of $H^1_{\mathcal{M}}(A^{(\alpha)})$. In that case the fermionic integral cannot be saturated, and the contribution to the partition function vanishes. This agrees with the fact that non-trivial $\mathrm{Ferm}(H^1_{\mathcal{M}}(A^{(\alpha)}))$ corresponds to a zero mode in the operator $L_-^{(\alpha)}$ in (2.6). We shall encounter such an example in the study of Seifert manifolds.

Suppose now we consider a link L in S^3. As discussed in refs. [16] and [17] using skein relations as well as surgery and S and T matrices, the U(1, 1) WZW model should give rise to its multivariable Alexander polynomial. After taking into account the proper Chern–Simons WZW correspondence, we expect a similar result to hold for U(1, 1) Chern–Simons theory. Let us comment on this, using now a purely three-dimensional point of view.

Recall first that the definition of the Alexander polynomial used here is the one adapted to the multivariable case, i.e., we divide the generator of the Alexander ideal by $t-1$. One has to set $t=\exp(-2i\pi e/k)$ to match with quantum field theory results [#2]. The knot complement has $H_1(S^3-K)=Z$ and the variable t of the Alexander polynomial corresponds to the generator of this H_1 for algebraic computation of torsion. Call Π the infinite cyclic multiplicative group with generator t. Recall the result of Milnor [19] that the Alexander polynomial of a link in S^3 is equal to the "algebraic" torsion of its complement (both objects being defined modulo Π here).

As far as analytic torsion is concerned let us go back to the Chern–Simons point of view. We can cut out a tubular neighborhood of the knot K and compute the functional integral in two steps. First we integrate over the interior of the solid torus, which results in a wave function for the gauge field A at its boundary. Choose coordinates such that dx^1 is along the meridian, dx^2 is along the longitude, dx^0

[#2] The t used here is equal to the square of the variable t in ref. [17].

points inward. The holonomy of A^1 being fixed [26] (and determining the flat connection of the complement) we get the condition that the component a^1 of the fluctuating part of one-forms has to vanish at the boundary. To insure that L_- is antihermitean and Δ is hermitean, we ask also that zero forms vanish at the boundary. In the exterior the argument leading to (2.6) and its exactness can be applied. We get therefore the appropriate ratio of products of determinants subject to these boundary conditions. Unfortunately they do not reproduce the "absolute" or "relative" boundary conditions used in ref. [10], for which the equality of analytic and algebraic torsion was proven [11], but rather are a mixture of them. We suspect, however, that the ratio of determinants in the analytic torsion is not too sensitive to boundary conditions. As an argument in that direction, notice, as proven in ref. [19], that the chain complex $Q(t)\otimes_{\Pi} C_*(\tilde{K}, Z)$ (where $Q(t)$ is the field of rational functions in t over rational numbers and $\tilde{K}$ is the infinite cyclic covering of the complement), is acyclic. Accordingly the algebraic torsion [10] is well defined modulo Π, independently of the metric chosen on S^3-K [10]. Also the algebraic torsion of the boundary is equal to one modulo Π [19]. Therefore following ref. [10] we expect that the analytic torsions for relative or absolute boundary conditions should be equal up to powers of t.

It would be interesting to clarify the above discussion, in particular concerning the role of Π. If we assume that boundary conditions inherited from Chern–Simons indeed lead to the Milnor torsion, we see that the Milnor theorem is equivalent to the exactness of the one-loop computation in U(1, 1) Chern–Simons theory.

3. Torsion of Lens spaces and Seifert manifolds

We compute in this section the torsion of some Lens spaces and Seifert manifolds using surgery and the S and T matrices of the U(1, 1) Wess–Zumino model. Due to the one-loop exactness, the torsion is extracted at finite k.

Flat connections that are not pure A_N give a finite contribution to the partition function, but we do not know how to extract it in a rigorous fashion from the above infinite background. There is, however, a well-defined procedure that gives correct results; it consists in computing invariants of the manifold with special links inside that act as "observers" and factor out the pure A_N flat connections. To explain this let us first consider the example of the Lens space $L(q, p)=X(p/q)$. Let us put a knot K in it as follows: we start with $S^2\times S^1$, remove a solid torus $D\times S^1$ that contains a Wilson loop carrying the representation (en/I), and glue it back after twisting the boundary by a SL(2, Z) matrix with first column $\binom{p}{q}$. Suppose moreover p/q has the simple continued fraction expansion

$$p/q=a_2-1/a_1\,, \tag{3.1}$$

and we suppose $a_1=q$ for simplicity (like in $L(q,-1)$, otherwise take complex conjugates of the final expressions). To compute the corresponding invariant we follow the general strategy, i.e., apply $T^{a_2}ST^{a_1}S$ to $|en/I\rangle$ and evaluate the scalar product of the final state with $|n=0\rangle$. Recall the U(1, 1) S matrix elements

$$S_{en/I}^{e'n'/I}=-\mathrm{i}V\exp\{-(2\mathrm{i}\pi/k)[e'(\tilde{n}-1/2)+e(\tilde{n}'-1/2)]\}, \qquad (3.2)$$

where $\tilde{n}=n+(e/2k)$, $V=1/(2k)$,

$$S_{en/I}^{\hat{n}'}=V\frac{\exp(-2\mathrm{i}\pi en'/k)}{2\sin(\pi e/k)}. \qquad (3.3)$$

Recall also the T matrix

$$T_{en/I}^{en/I}=\exp 2\mathrm{i}\pi[(\tilde{n}-1/2)e/k+\tfrac{1}{12}]. \qquad (3.4)$$

One finds [17]

$$\mathscr{Z}(L(q,p),K)=-\mathrm{i}V^2\exp[\tfrac{1}{6}\mathrm{i}\pi(a_1+a_2)]$$
$$\times\sum_{n'}\sum_{e'\neq 0}\frac{\exp\{(2\mathrm{i}\pi/k)[qe'(\tilde{n}'-1/2)-e'(\tilde{n}-1/2)-e(\tilde{n}'-1/2)]\}}{2\sin(\pi e'/k)}. \qquad (3.5)$$

The result depends therefore on the number of solutions in the fundamental domain [17] of the equation (for the unknown e')

$$qe'=e \bmod 2k. \qquad (3.6)$$

If there is no solution the invariant vanishes.

It is instructive to interpret the above result geometrically. The empty Lens space has $\Pi_1=\mathrm{Z}_q$ with a generator g that satisfies $g^q=1$. If we consider now the complement of the above knot K in the Lens space (obtained by removing a tubular neighborhood of K) it is easy to see that it has $\Pi_1=\mathrm{Z}$ with generator t satisfying $g^q=t$. This complement is in fact the second solid torus in the surgery prescription, i.e., it is the same as for a simple loop S^1 embedded in S^3. However, due to the twisting of the boundary, "from the S^3 point of view" the generator of the Π_1 of the complement is g. Hence the invariant should be obtained by taking the formula for the invariant in S^3 but replacing the e charge by $e'=e/q \bmod 2k$, as we indeed find. The phase term comes precisely from the twisting of the boundary of the torus which turns into a non-trivial self-framing here. In a "classical" theory of the Alexander polynomial, the invariant of K in the Lens space would not depend on arithmetic properties as (3.6). In the quantum theory, however, charge quantization can make this invariant vanish.

The above result is recovered using the classical computations of Alexander invariants [14]. Represent the Π_1 of the knot complement as

$$\{t,g:tg^{-q}=1\}. \qquad (3.7)$$

Then the matrix of Fox derivatives is [27]

$$\| t^{-1}, \ -t(t^{-1}-1)/(t^{-1/q}-1) \| \,, \tag{3.8}$$

so the Alexander polynomial is, up to a power of t,

$$\Delta_{\text{Alex}} \propto 1/(t^{1/q}-1) \,. \tag{3.9}$$

Notice that there is an ambiguity if we want to give a "numerical value" to this Alexander invariant, since several roots of t can be chosen. This is manifest in (3.5), where the expression in the right hand side is periodic in e mod $2k$, and can be interpreted again as a quantum effect.

Of course our purpose is not so much to compute invariants of links in manifolds but to extract from them properties of the manifold itself, using the link as an "observer", and letting ultimately the E charge carried by its Wilson loop go to zero (mod $2k$). Notice that, strictly speaking, this procedure is not allowed in the quantum field theory, where charges take discrete values and are limited to the fundamental domain. Let us clarify this. Suppose there is a single solution to (3.6): $e'=e/q$. Then the invariant is

$$\begin{aligned} \mathscr{Z}(L(q,p),K) = & -\mathrm{i}V \exp[\tfrac{1}{6}\mathrm{i}\pi(a_1+a_2)] \\ & \times \frac{\exp[-(2\mathrm{i}\pi/k)(e/q)(\tilde{n}-1/2)]}{2\sin(\pi e/kq)} \,. \end{aligned} \tag{3.10}$$

Let us take now the (formal) limit $e=e_0 \to 0$. One finds then

$$\mathscr{Z}(L(q,p))^{(0)} \to -\exp[\tfrac{1}{6}\mathrm{i}\pi(a_1+a_2)] \frac{\mathrm{i}q}{4\pi e_0} \,. \tag{3.11}$$

This agrees with the computation of the invariant of an "empty" Lens space [17]: it is the order of the homology, q, up to a framing factor and a numerical factor that coincides with the invariant of S^3. The limit (3.11) is of course determined by pure A_N flat connections only. Suppose now we take the (formal) limit $e=e_0+2fk$ with $e_0 \to 0$ and $f=0, ..., q-1$. In that case one has

$$\begin{aligned} \mathscr{Z}(L(q,p),K)^{(f)} = & -\mathrm{i}V \exp[\tfrac{1}{6}\mathrm{i}\pi(a_1+a_2)] \\ & \times \frac{\exp[-(2\mathrm{i}\pi(e_0+2fk)/kq](\tilde{n}-1/2)]}{2\sin[\pi(e_0+2fk)/kq]} \,. \end{aligned} \tag{3.12}$$

This has a finite limit for $f \neq 0$. Indeed the E charge of the Wilson loop constrains the holonomy t and pure A_N flat connections are projected out. As we explain next, the limit is then the Milnor torsion of the appropriate flat connection.

To deal with a slightly more complicated case consider the Lens space $L(p, -q)$ with p/q the same as in (3.1). We now put in L two Wilson lines in the following fashion. We start with $S^2 \times S^1$, remove a solid torus $D \times S^1$ where we put a loop carrying some representation $|en/I\rangle$, and glue it back after twisting the second $D\times S^1$ where we inserted a loop carrying $|e'n'/I\rangle$. The partition function is obtained by taking the scalar product of $ST^{a_2}ST^{a_1}S|en/I\rangle$ with $|e'n'/I\rangle$. Only two-dimensional representations appear now in the intermediate states and the computation looks much like in the U(1) case. Summing over intermediate states gives rise to arithmetic constraints as before. Solve these constraints formally by taking their simplest solution as above, this gives, up to framing of the manifold and factors depending on k only,

$$\exp\left\{\frac{-2i\pi}{k(a_1a_2-1)}\left[a_2e(\tilde{n}-1/2)+a_1e'(\tilde{n}'-1/2)+e'(\tilde{n}-1/2)+e(\tilde{n}'-1/2)\right]\right\}. \tag{3.13}$$

Since $p=a_1a_2-1$, $q=a_1$ we can set $a_2=q^*$ with

$$q^*q=1 \bmod p. \tag{3.14}$$

From now on we set $e'=0$. As before we set $e=e_0+2fk$, $b=0, ..., p-1$ and we let $e=e_0\to 0$ to extract properties of the Lens space itself. We get, collecting all factors,

$$\begin{aligned}\mathscr{Z}(L(p,q),K_1,K_2)^{(f)}=&\,iV\exp[\tfrac{1}{6}i\pi(a_1+a_2)]\\ &\times\exp\frac{-4i\pi q^*f^2}{p}\\ &\times\exp\left\{\frac{-2i\pi}{p}\left[(2n-1)q^*f+(2n'-1)f\right]\right\}.\end{aligned} \tag{3.15}$$

In the first exponential we recover the Chern–Simons action of the U(1) flat connection associated with the representation of $\Pi_1=Z_p$:$g=\omega^f$, $\omega=\exp(2i\pi/p)$. The E charge being 0 or a multiple of $2k$ the Wilson loops behave as observers and simply project onto a particular flat connection of the Lens space; the second exponential then measures their holonomies. For simplicity we have factored onto flat connections with trivial A_N part; it suffices to consider $n=n_0+f'k$ to extract the remaining ones, whose contributions differ from (3.15) by phase factors only (their torsion is the same).

Now (3.15) is actually the partition function of the particular flat connection times the product of two traces, one for each Wilson loop. We therefore deduce the contribution of the flat connection itself,

$$iV\exp[\tfrac{1}{6}i\pi(a_1+a_2)]\exp\frac{-4i\pi q^*f^2}{p}\frac{1}{4\sin(2\pi f/p)\sin(2\pi q^*f/p)}. \tag{3.16}$$

The factor V occurs, as explained earlier, as the inverse of the volume $U(1)_E\times U(1)_N$. The first exponential occurs from the framing of the Lens space and the i from spectral flow [7]. The second exponential gives the Chern–Simons action [there is no k factor here due to the form of (2.7)]. Forgetting for a while the torsion of the trivial representation and comparing with (2.6), we conclude that the fraction is equal to the Milnor torsion of the flat connection,

$$\tau^{(f)}=1/|\omega^{2f}-1|\,|\omega^{2q^*f}-1|\,, \tag{3.17}$$

in agreement with the known results [28]. A similar result follows from the above calculation for $L(q, p)$ above. In that case, since only one loop has been inserted in the Lens space, one has to divide only by one trace. However, the other sin term has already been provided by the S matrix element, as is usually the case for computations of Alexander invariants. Finally the computation can be generalized to arbitrary continued fraction expansion as in ref. [17] with similar results.

Dividing by the traces of the Wilson loops has also a torsion interpretation. Indeed, in our case we have the factorization $\tau(M)=\tau(M-K^{\mathrm{b}})\tau(K^{\mathrm{b}})$ [19,29] (where we used the fact that the torsion of the boundary is one) and K^{b} is a tubular neighborhood of K. $\tau(K^{\mathrm{b}})$ is the torsion of a solid torus, which is also the complement of S^1 into S^3, and for which we know that the torsion (up to phase factors) is equal to $\Delta_{\mathrm{Alex}}\propto 1/(t-1)$.

The complete comparison with (2.6) involves an additional factor equal to the torsion of the trivial flat connection, well defined only after a homology basis is chosen. With the usual choice, $\tau_0=p$ [7] here. We are missing this term in (3.16) because we did not treat the arithmetic constraints in the most correct fashion.

Let us now remedy this. We first consider for a while the case of U(1). For level κ we have the S and T matrices

$$S_{ab}=\frac{1}{\sqrt{\kappa}}\exp(-2\mathrm{i}\pi ab/\kappa)\,, \qquad T_{aa}=\exp \mathrm{i}\pi(a^2/\kappa-\tfrac{1}{12})\,, \tag{3.18}$$

where $a, b=0, ..., \kappa-1$ and κ is even. Notice that these matrix elements are invariant under translations of a, b mod κ. Let us now compute the invariant of the Lens space $L(p, -1)$ with two Wilson lines carrying representations $|a\rangle$ and $|c\rangle$ as above. We have

$$\mathscr{Z}(L(p, -1), K_1, K_2)=\exp(-\mathrm{i}\pi p/12)$$
$$\times\frac{1}{\kappa}\sum_{b=0}^{\kappa-1}\exp\left[\frac{\mathrm{i}\pi}{\kappa}(-2ab+pb^2-2bc)\right]. \tag{3.19}$$

We would like to perform the resummation of (3.19) in a formal way, without using finite Gaussian sum formulas [8]. Due to the translation invariance of the exponent in (3.19) we can extend the summation for b over the entire set Z. Let us also sum over a mod κ. We write symbolically (we suppress for a while the framing factor, which is all that is implied in the symbol $\propto$)

$$\mathscr{Z}(L(p,-1),K_1,K_2) \propto \frac{1}{\kappa}\frac{1}{\Lambda^2} \times \sum_{n_a\in Z}\sum_{b\in Z} \exp\left[\frac{i\pi}{\kappa}(-2ab+pb^2-2bc)\right]\exp(-2i\pi n_a b), \quad (3.20)$$

where Λ are "normalization factors". The sum over n_a, however, due to the last exponential, constrains by itself b to be an integer. We can therefore replace the second sum by an integral, dropping at the same time one of the normalization factors,

$$\mathscr{Z}(L(p,-1),K_1,K_2) \propto \frac{1}{\kappa}\frac{1}{\Lambda} \times \sum_{n_a\in Z}\int \mathrm{d}b \exp\left[\frac{i\pi}{\kappa}(-2(a+\kappa n_a)b+pb^2-2bc)\right]. \quad (3.21)$$

Compute now the Gaussian integral over b:

$$\mathscr{Z}(L(p,-1),K_1,K_2) \propto \frac{i^{1/2}}{\sqrt{\kappa p}}\frac{1}{\Lambda}\sum_{n_a\in Z}\exp\left[\frac{-i\pi}{\kappa p}(a+c+\kappa n_a)^2\right]. \quad (3.22)$$

The exponential in (3.22) is invariant under shifts of n_a mod p so we can split the sum over n_a as a sum over f and a sum over n with $n_a = f + np$. The sum over n produces a multiplicity cancelling the normalization factor so we get

$$\mathscr{Z}(L(p,-1),K_1,K_2) \propto \frac{i^{1/2}}{\sqrt{\kappa p}}\sum_{f=0}^{p-1}\exp\left[\frac{-i\pi}{\kappa p}(a+c+\kappa f)^2\right]. \quad (3.23)$$

Let now $a=c=0$ to get (after reinstalling the framing factor)

$$\mathscr{Z}(L(p,-1)) = \exp(-i\pi p/12)\frac{i^{1/2}}{\sqrt{\kappa p}}\sum_{f=0}^{p-1}\exp\left(\frac{-i\pi\kappa f^2}{p}\right). \quad (3.24)$$

This result can of course be recovered by a correct treatment of the "normalization factors", i.e., by regularizing the sums. The interpretation of (3.23) is straightforward. We have again a sum over U(1) flat connections. The determinants are not twisted so as a global factor we get the inverse square root of the torsion of the trivial flat connection. The factor of κ as above comes from the volume of the group, since connections are reducible. We do not have to divide by the contribution of the inserted loop in that case, since it is simply equal to one, so we can suppress the label K in the final result (3.24).

This method reproduces correct results in the SU(2) case as we demonstrate now. Consider again the Lens space $L(p,-1)$ with a Wilson line carrying spin j (in units where spins are integer). We find

$$
\begin{aligned}
\mathscr{Z}(L(p,-1),K)=&\left(\frac{\mathrm{i}}{2p(k+2)}\right)^{1/2}\exp(-\tfrac{1}{4}\mathrm{i}\pi p)\\
&\times\sum_{f=0}^{p-1}\exp[-2\mathrm{i}\pi(k+2)f^2/p]\left[\cos(2\pi fj/p)\exp\left(\frac{-\mathrm{i}\pi j^2}{2(k+2)p}\right)\right.\\
&\left.-\cos(2\pi f(j+2)/p)\exp\left(\frac{-\mathrm{i}\pi(j+2)^2}{2(k+2)p}\right)\right].
\end{aligned}
\tag{3.25}
$$

If we set $j=0$ we recover the result of ref. [8] (up to framing, for which we have put no corrections here). Let us instead consider the limit $k\to\infty$. We get then

$$
\begin{aligned}
\mathscr{Z}(L(p,-1),K)\simeq&\left(\frac{\mathrm{i}}{2pk}\right)^{1/2}\exp(-\tfrac{1}{4}\mathrm{i}\pi p)\\
&\times\sum_{f=1}^{[(p-1)/2]}\exp[-2\mathrm{i}\pi kf^2/p]\,4\sin(2\pi f/p)\sin(2\pi f(j+1)/p).
\end{aligned}
\tag{3.26}
$$

The first sin in this formula is the inverse of the torsion of the complement. The second term can be explained as before. It arises since the Wilson line creates some "small" (of order $1/k$) background field through the holonomy condition. This field couples with the "strong" (of order k^0) flat connection field proportional to f so that they contribute a bilinear term in the Chern–Simons action. Due to the overall normalization in this action, the final contribution does not depend on k. The analog in the U(1, 1) case was a simple exponential: we get here a sin due to Weyl reflection, so in fact we have to deal with two holonomies of opposite sign (see, e.g., ref. [26]). To get the contribution to the partition function of the manifold itself we have to divide this result by the contribution of the loop, which is the character of the spin j representation evaluated at the corresponding value of the holonomy,

$$
\mathrm{ch(holonomy)}=\frac{\sin(2\pi f(j+1)/p)}{\sin(2\pi f/p)}.
\tag{3.27}
$$

We thus get in the end the contribution of the flat connection to be $\sin^2(2\pi f/p)$ as expected since the usual variable $q^*=1$ here [#3]. The sum in (3.26) runs only from 1 to the integer part of $(p-1)/2$ due to "folding" of flat connections by Weyl reflection. Notice the case $f=p$ does not contribute at this order. This is because for $f=p$ one has to divide by the entire volume of the group, which contributes additional factors of k in the denominator. Appropriate factors of p appear also in that case. The torsion is also obtained from (3.25) or (3.26) by letting $j=0$. The point is that we cannot put directly the identity representation in the Alexander case since then one gets an expression dominated by pure A_N

[#3] This is the inverse of the Milnor torsion with our definitions.

flat connections.

We can now get back to U(1, 1). The above resummation method applied formally in that case does not run into the arithmetic constraints of the above direct calculation; it can be considered as an ansatz to extract the finite contributions to the partition function of manifolds. In the case of the Lens space $L(p, q)$ we reproduce with this ansatz the result (3.16) together this time with the contribution of the trivial connection. Let us rather discuss the more interesting case of a Seifert manifold $X(p_1/q_1, p_2/q_2, p_3/q_3)$. It is obtained from $S^2 \times S^1$ by obvious generalization of the construction of the Lens space $L(p, q) = X(q/p)$. Recall that the Π_1 has a presentation $\{h, g_1, g_2, g_3;\ x^{p_i} h^{q_i} = 1,\ x_1 x_2 x_3 = 1,\ h \text{ central}\}$, while $\mathrm{order}(H_1(X)) = p_1 p_2 p_3 + \text{permutations}$ (we assume for simplicity it is positive, otherwise one has to take the absolute value). For further simplicity we restrict to $q_1 = q_2 = q_3 = 1$. Put in each of the three tori $D^2 \times S^1$ a Wilson loop carrying a representation $(e_i n_i/I)$. Put on the last torus (en/I). We compute the invariant by acting on each state with $T^{p_i}S$. As before, constraints are met, which, when solved formally, would give (we suppressed for a while framing factors)

$$\mathscr{Z}(X(p_1, p_2, p_3), L) \propto \mathrm{i}V \exp\left\{-\frac{2\mathrm{i}\pi}{k}\left[\frac{p_1 p_2 p_3}{p_1 p_2 + p_1 p_3 + p_2 p_3}\left(\sum_i \frac{e_i}{p_i} + e\right)\right.\right.$$
$$\left.\left.\times\left(\frac{\sum_i \tilde{n}_i - 1/2}{p_i} + n - 1/2\right) - \sum_i \frac{e_i(\tilde{n}_i - 1/2)}{p_i}\right]\right\}$$
$$\times \sin^2\left[\frac{2\pi}{k}\frac{p_1 p_2 p_3}{p_1 p_2 + p_1 p_3 + p_2 p_3}\left(\sum_i \frac{e_i}{p_i} + e\right)\right]. \tag{3.28}$$

The detailed structure of Π_1, i.e. of the flat connections, depends now on the arithmetic properties of p_i. Setting generally the e_i to be appropriate multiples of $2k$, we shall read the Chern–Simons action and the holonomies in the above exponential. The torsion will be obtained by dividing by a product of traces for each Wilson loop, hence will have the form of $\tau = \sin^2/\sin^4$, as is known. To get further results we can apply the resummation ansatz. After computations one finds for instance that, if $p_2 + p_3$ and $p_1 p_2 + p_1 p_3 + p_2 p_3 = P$ are coprimes, and if we set $e_2 = e_3 = e = 0$,

$$\mathscr{Z}(X(p_1, p_2, p_3), L) = \frac{\mathrm{i}V}{P}\exp\left[\tfrac{1}{6}\mathrm{i}\pi(p_1 + p_2 + p_3)\right]$$
$$\times \sum_{f=0}^{P-1} \exp\left(\frac{-4\mathrm{i}\pi}{P}(p_2 + p_3) f^2\right) 4\sin^2\left(\frac{2\pi}{P} p_2 p_3 f\right), \tag{3.29}$$

which should completely describe the structure of flat connections in that case, with torsion

$$\tau^{(f)} = \left| \frac{\sin(2\pi p_2 p_3 f/P)}{4\sin(2\pi(p_2+p_3)f/P)\sin(2\pi p_2 f/P)\sin(2\pi p_3 f/P)} \right| , \qquad (3.30)$$

while the exponential gives the Chern–Simons action and the prefactor the torsion of the trivial representation equal to order$(H_1(X))=P$. Notice that (3.30) can sometimes vanish, indicating the existence of a zero mode of $L_-^{(\alpha)}$. We have not found (3.30) in the literature. It is compatible with the torsions computed in ref. [30] and with various limiting cases.

4. Manifolds with an infinite H_1

Using S and T matrices, invariants of manifolds with an infinite H_1 have also been computed in ref. [17]. The simplest case was

$$\mathscr{Z}(S^2\times S^1)=0 \qquad (4.1)$$

[this is obtained before dividing by $\mathscr{Z}(S^3)$]. The meaning of (4.1) in terms of conformal blocks was discussed in ref. [17]. We can now recover this result from the torsion point of view by considering $S^2\times S^1$ as the particular Lens space $L(0,1)$. Since $H_1(L)=Z$, there is a continuum of flat connections parametrized by their holonomy along a non-contractible cycle,

$$h=\exp[i(En+Ne)] , \qquad (4.2)$$

where E and N are the U(1, 1) generators defined in (2.3), while

$$\frac{2\pi e}{k} = \oint A_E^{\mu}\,dx_\mu , \qquad \frac{2\pi n}{k} = \oint A_N^{\mu}\,dx_\mu . \qquad (4.3)$$

The corresponding torsion is

$$\tau^{(en)} = \frac{1}{4\sin^2(\pi e/k)} . \qquad (4.4)$$

Note that the manifold $S^2\times S^1$ can be formed by glueing together two solid tori, so τ is the product of their torsions.

The invariant of $S^2\times S^1$ is proportional to the integral of the torsion over the moduli space of flat connections:

$$\mathscr{Z}(S^2\times S^1) \propto \int_0^{2\pi} dx \frac{1}{\sin^2(x/2)} . \qquad (4.5)$$

Such an integral appeared in ref. [17] in the expression for the volume of U(1, 1). We concluded there that this integral when properly regularized has to be set equal to zero, hence reproducing (4.1).

The integrand in (4.5) is the square of the "denominator" in the Weyl character formula. It is also the Jacobian factor for switching from the integral over the group to the integral over its maximal torus. This is not a coincidence: for a general bosonic Lie group G the torsion for $S^2 \times S^1$ is

$$\tau_G^{(\lambda)} = \prod_{\alpha>0} |e^{i\alpha\cdot\lambda/2} - e^{-i\alpha\cdot\lambda/2}|^2 , \tag{4.6}$$

where $\lambda = \oint A^\mu \, dx_\mu$ belongs to the Cartan subalgebra. Indeed, the one-loop approximation of the Chern–Simons path integral over $S^2 \times S^1$ with two Wilson lines carrying representations i and j of G along the non-contractible cycle, expresses orthonormality of Kac–Moody characters [2] by

$$\int_{mt/W} d\lambda \, \tau_G^{(\lambda)} \chi_i(\lambda) \chi_j(\lambda) = \delta_{i\bar{j}} , \tag{4.7}$$

where mt/W denotes the maximal torus factored over the action of the Weyl group, and $\chi_{i,j}(\lambda)$ are the characters of the representations i and j.

More generally, it is interesting to consider the Milnor torsion of the manifold $X_h \times S^1$ and compare it with the formula for the invariant obtained in ref. [17]. By glueing and using the basic formula of the torsion of a solid torus, one finds first of all [25,30]

$$\tau^{(en)}(X_h \times S^1) = |2 \sin(\pi e/k)|^{2h-2} . \tag{4.8}$$

The one-loop approximation to the path integral, which we expect to be exact here, involves an integral over the moduli space of flat connections. We assume that this can be simply computed by integrating (4.8) over e, the ratio of determinants in (2.6) becoming a measure on the moduli space [22]. One finds then

$$\int \tau^{(en)} = 2^{2h-2} \sum_{n=0}^{2h-2} \binom{n}{2h-2} \int_0^{2\pi} e^{ix(h-1-n)} \frac{dx}{2\pi}$$

$$= 2^{2h-2} \binom{h-1}{2h-2} . \tag{4.9}$$

This binomial coefficient is easily interpreted in terms of conformal blocks. It is precisely the invariant obtained in ref. [17], up to some statistics factor. This additional factor arose in ref. [17] due to the splitting in the quantization of the WZW model $|en\rangle \to |en/1\rangle$, $|en/2\rangle$. It is a puzzling problem about the WZW–CS correspondence to interpret such splitting from the three-dimensional point of view.

5. Conclusions

We finally would like to comment about U(1, 1) versus GL(1, 1). In many respects, these two groups differ as their respective bosonic parts $U(1) \times U(1)$

and $R \times R$ do. Their representations are similar; in the former case the numbers e, n are quantized, while they take continuous values in the latter. Both groups have a trivial Π_3 so the level k is not quantized in the Wess–Zumino theories. Although the bosonic part of U(1, 1) is compact, in both cases the corresponding part in the Wess–Zumino action has indefinite metric with signature $(1, -1)$. Both theories lead to a spectrum unbounded from below. Most computations in ref. [16], in particular the computation of four-point functions, apply to both cases. S and T matrices have similar properties, although for GL(1, 1) one has to integrate over e, n in surgery computations (the treatment of indecomposable and one-dimensional representations is then quite subtle). Strictly speaking, the end of section 5 and section 7 in ref. [16] apply to U(1, 1) only (this is not indicated in the published version). Considering now Chern–Simons theories, both groups lead to the same invariant, the Alexander polynomial, for links in S^3. Their difference appears when dealing with invariants of links in more complicated manifolds. For empty manifolds in particular, there are $(\text{order}(H_1))^2$ flat connections for U(1, 1), but only the trivial one for GL(1, 1). Up to the respective volumes of the groups, invariants are equal to $\text{order}(H_1)$ in the former case, to 1 in the latter.

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